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PRECIPITOUS IDEALS AND Σ_4^1 SETS

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ABSTRACT

We prove under the assumption of the existence of a measurable cardinal and precipitous ideal on ω_1, that every Σ_3^1 set is Lebesgue measurable, has the Baire property and is either countable or contains a perfect subset. We get similar results for Σ_4^1 sets, if we add the additional assumptions of C. H. and that $P_\omega(2^{2^{\omega_1}})$ carries a normal precipitous ideal.

0. Introduction

Many problems concerning the projective hierarchy of sets of reals turned out to be independent of the usual axioms of Set Theory. A typical problem is: "What is the "simplest" set of reals which is not Lebesgue measurable or does not have the Baire property", and such problems need some additional set theoretical assumptions for being settled.

In this paper we present a case in which combinatorial assumptions on the existence of certain kinds of ideals introduced by T. Jech and K. Prikry [5] (precipitous ideals) can given some results on the behaviour of Σ_3^1 and Σ_4^1 sets of reals. The properties of sets of reals in which we shall be interested are Lebesgue measurability, Baire property (i.e., being equal to an open set up to a set of the first category) and being either countable or containing a perfect subset. Some results about partitioning ω_1 into two large sets will be also given.

Our notation is rather standard. (See Jech [4] for reference.) As usual in Descriptive Set Theory reals will be identified with members of the space ω^ω. For basic facts about the Σ_n^1 hierarchy see also Sheonfield [15]. Note that in this paper Σ_n^1 sets are boldface, i.e., can contain real parameters in their definition. We also assume acquaintance with basic forcing techniques. When we claim that a statement is being forced we mean that it is forced by every condition. The ground model will be usually V and $V[G]$ is the generic extension using the

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generic filter G. Unless otherwise stated lower case $x, y, z, \cdots$ denote reals (i.e., members of ω^ω), lower case Greek letters denote ordinals. The noteworthy exception is when they denote terms in the forcing language for some forcing notion. (Note that every member of $V[G]$ is the realization with respect to G of some term which lies in V, see [16].) $P(A)$ is the power set of A. $[A]^n$ is the set of all subsets of A of cardinality n. When we would like to relativize a given notation to a certain universe of set theory we use superscripts. Thus $P^V(A)$ is the power of A in the sense of V, ω_1^V is the first uncountable ordinal in V, etc. $A \Delta B$ is the symetric difference between A and B.

For $x \in \omega$, $x \restriction n$ is the restriction of x to $n = \{0, \cdots, n-1\}$. Given pair of reals $\langle x, y\rangle$ we code the pair as one real $x * y$ by defining $x * y(2n) = x(n)$, $x * y(2n+1) = y(n)$. Seq(A) is the set of all finite sequences of members of A. Seq(A) naturally forms a tree where the tree partial order is the extension relation between finite sequences. We naturally identify Seq$(A \times B)$ with Seq$(A) \times$ Seq(B). We fix an enumeration of Seq(ω), $s_0, \cdots, s_i, \cdots$, such that length of $s_i \leqq i$ and such that every initial segment of s_i appears before s_i in the enumeration. The length of $s \in$ Seq(A) is $l(s)$.

For a tree T which is a subtree of Seq$(\omega \times A)$, $x \in \omega^\omega$, define T_x to be the subtree of T defined as the set: $\{< x \restriction n, f \restriction n\rangle \mid \langle x \restriction n, f \restriction n\rangle \in T\}$.

It is classically known that every Σ_1^1 set is Lebesgue measurable, has the Baire property, and is either countable or contains a perfect subset [10]. Solovay in [18] proved that the same applies to Σ_2^1 sets if one assumes that there are only countably many reals constructible from a given real. This last condition follows from the existence of measurable cardinal. Mansfield in [13], assuming the existence of measurable cardinal, constructed for every Σ_3^1 set an ordinal definable tree T such that if in $L[T]$ there are only countably many reals then every Σ_3^1 set is Lebesgue measurable, has the Baire property, and is either countable or contains a perfect subset. In fact our proofs will rely very heavily on Mansfield's results and for Σ_3^1 sets they will be basically showing that under our assumption Mansfield's condition holds. Solovay in [19] showed the existence of a model of Set Theory in which every Σ_n^1 (in fact, every real ordinal definable set) has all the properties we mentioned. Here we shall be more interested in implications between combinatorial assumptions and facts about Σ_3^1 and Σ_4^1 sets than in consistency results.

All ideals and filters will be nonprincipal.

For the readers convenience we reproduce the definition of precipitous ideal. For an ideal $\mathcal{I}$ on a set I, $B \subseteq I$ is said to be of positive measure with respect to $\mathcal{I}$ if $B \notin \mathcal{I}$. In the above definition we omit "with respect to $\mathcal{I}$" whenever $\mathcal{I}$ is well understood from the context. $\mathcal{P}(\mathcal{I})$ is the collection of sets positive with respect to $\mathcal{I}$ partially ordered by inclusion.

Definition 0.1. ([5], [6]) $\mathcal{I}$ is an ideal on a set I. Define the infinite game $\mathcal{G}(\mathcal{I})$ as follows. We have two players picking alternately members of $\mathcal{P}(\mathcal{I})$ such that each set picked is a subset of the last set picked so far. Thus they construct an infinite decending sequence of members of $\mathcal{P}(\mathcal{I}) \cdots, A_{n+1} \subseteq A_n \cdots$. Player I wins if $\bigcap_{n<\omega} A_n = \emptyset$. Player II wins otherwise. The ideal $\mathcal{I}$ is said to be precipitous if player I has no winning strategy in the game $\mathcal{G}(\mathcal{I})$.

Note that a precipitous ideal must be ω_1 complete otherwise if $A \in \bigcup_{n<\omega} I_n \not\in \mathcal{I}$ where $\forall n < \omega$, $I_n \in \mathcal{I}$, then player I can easily win the game by playing A as his first move and then at his n-th turn there after playing $B_{n-1} - I_n$ where B_{n-1} is the last move made by his opponent.

The next lemma will provide an equivalent definition to precipitous ideals, but before the lemma we need some remarks. Suppose that we use $\mathcal{P}(\mathcal{I})$ as a set of forcing conditions. A generic filter in $\mathcal{P}(\mathcal{I})$ will be clearly an ultrafilter in the Boolean Algebra $P^V(I)$ (the power set of I in the sense of V) which extends the dual filter of I. Given such an ultrafilter U, we can form the ultrapower $\mathrm{Ult}(V, I, U)$ where its elements are functions from I into V which lie *in* V reduced by the equivalence relation $\equiv_U$ (i.e. $f \equiv_U g$ if $\{i \mid f(i) = g(i)\} \in U$). The equivalence class of a function $f: I \to V$ modulo U will be denoted by $[f]_U$, where we may drop the subscript U, if understood from the context. We use this notation even when U is just a filter.

Though this is not the usual ultrapower construction, Łos theorem [12] still applies (using the fact that V is a model of ZFC) and $\mathrm{Ult}(V, I, U) \models \Phi([f_1], \cdots, [f_n])$ iff $\{i \mid V \models \Phi(f_1(i), \cdots, f_n(i))\} \in U$.

Lemma 0.2. ([2]) *$\mathcal{I}$ is precipitous if and only if for every U, a V generic filter in $P(\mathcal{I})$,* $\mathrm{Ult}(V, I, U)$ *is well founded.*

In case $\mathrm{Ult}(V, I, U)$ is well founded, it is isomorphic to a transitive class M_U. Note that V can naturally be elementarily embedded in $\mathrm{Ult}(V, I, U)$, hence in M_U. We are going to confuse $\mathrm{Ult}(V, I, U)$ and M_U, hence equivalence classes of functions in V^I and members of M_U. Similarly i_U will denote simultaneously the embedding of V into $\mathrm{Ult}(V, I, U)$ and into M_U. Again we usually drop the subscript U in i_U and in M_U.

The existence of a precipitous ideal on ω_1 is equiconstant with the existence of a measurable cardinal. (See [6].) Throughout this paper we assume the existence of a measurable cardinal. We usually fix one measurable cardinal denoted by κ.

Additional assumptions will be made explicitly in each of the theorems. The main theorems of this paper are (M.C. is the assumption of the existance of measurable cardinal):

THEOREM 1.4 (M.C.) *If ω_1 carries a precipitous ideal then every Σ^1_3 set is Lebesgue measurable, it has the Baire property, and is either countable or contains a perfect subset.*

For the next theorem we need the following terminology: $P_\mu(\lambda)$ is the set of all subsets of λ of cardinality $<\mu$; an ideal $\mathcal{I}$ over $P_\mu(\lambda)$ is called normal if

(a) it covers λ, i.e., for all $\alpha\in\lambda$, $\{P \mid \alpha\notin P, P\in P_k(\lambda)\}\in\mathcal{I}$,

(b) every choice function on a set in $\mathcal{P}(\mathcal{I})$ (i.e., a function f such that for P in its domain $f(P)\in P$) is constant on a subset of its domain which is still in $\mathcal{P}(\mathcal{I})$.

Note that μ can be considered to be a subset of $P_\mu(\mu)$ and for every normal ideal on $P_\mu(\mu)$, $P_\mu(\mu)-\mu$ is in the ideal, hence the ideal can be considered to lie on μ, and then it is normal in the usual sense, i.e., every function which is defined on a set in $\mathcal{P}(\mathcal{I})$ and which is regressive (i.e., $f(\alpha)<\alpha$ for all α in its domain) is constant on a set in $\mathcal{P}(\mathcal{I})$. The consistency of the existence of precipitous normal ideal on $P_{\omega_1}(\lambda)$ follows from the consistency of a super compact cardinal.

THEOREM 2.11. (M.C. + C.H.) *If ω_1 carries a precipitous ideal and $P_{\omega_1}(2^{2^{\omega_1}})$ carries a normal precipitous ideal then every Σ^1_4 set is Lebesgue measurable and has the Baire property.*

THEOREM 3.1. (M.C.) *Assume ω_1 carries a precipitous ideal and $P_{\omega_1}(2^{2^{\omega_1}})$ carries a precipitous normal ideal, then every Σ^1_4 set is either countable or contains a perfect subset.*

Section 4 contains some further results about splitting ω_1 into two "large" sets. Note that while in the proofs of these theorems we shall consider different universes of Set Theories (like generic extensions of the ground model V, etc.) the statement is always about the ground model which we usually denote by V.

1. Σ^1_3 sets

We review the characterization of Σ^1_3 sets essentially due to Martin and Solovay [14] which is modified and extensively exploited by Mansfield [13].

Let κ be a measurable cardinal and U a normal ultrafilter on κ. By forming the ultrapower $\mathrm{Ult}(U,\kappa,U)$ we get an elementary embedding j of V into the transitive isomorph of $\mathrm{Ult}(V,\kappa,U)$, $M_U=M_1$. j can be iterated by the usual techniques (see [8] and [1]) and the nth iterate of j, j_n, is an elementary embedding of a transitive class M_n into some subclass of it, M_{n+1}. In fact j_n can be described as follows. Let U_n be the ultrafilter on $[\kappa]^n$ defined by

$$A \in U_n \leftrightarrow \exists B \in U \quad [B]^n \subseteq A.$$

For 0, U_0 is the principle ultrafilter on $\{0\}$. Let $\mathrm{Ult}(V, [\kappa]^n, U_n)$ be the corresponding ultrapower which is well founded, hence isomorphic to some transitive class M_n. (Note that $M_0 = V$.) j_n is an embedding of M_n into M_{n+1} induced by the following embedding of $\mathrm{Ult}(V, [\kappa]^n, U_n)$ into $\mathrm{Ult}(V, [\kappa]^{n+1}, U_{n+1})$ defined by:

$$[f]_{U_n} \to [g]_{U_{n+1}} \qquad \text{where } g(\{\alpha_1 < \alpha_2 < \cdots < \alpha_{n+1}\})$$

$$= f(\{\alpha_1 < \cdots < \alpha_n\}).$$

For the rest of this paper we fix a measurable cardinal κ (which we assumed to exist) and a normal ultrafilter on it U. U_n and j_n are defined as above. Fix a given Σ_3^1 set A. A has the form

$$A = \{x \mid \exists z \forall z R(x, y, z) \quad \text{is not well founded}\},$$

where for every $x, y, z \in \omega^\omega$ $R(x, y, z)$ is a linear order of ω, such that $R(x, y, z)$ restricted to $n = \{0, \cdots, n-1\}$ depends just on $x \restriction n$, $y \restriction n$, $z \restriction n$. R can be considered to be a function from triples of $\mathrm{Seq}(\omega)$ into orderings of initial segments of ω, and can be coded as an element of ω^ω. Moreover given the definition of A as a Σ_3^1 set, R can be computed from this definition in a way which is absolute between models of Set Theory. We shall misuse the language by using the same notation A for the set $\{x \mid \Phi(x)\}$ (Φ is Σ_3^1 formula), in different models of Set Theory. The function R, since it just depends on the formula Φ and the parameters appearing in it, will be the same in all models we shall deal with.

Define π: $\mathrm{Seq}(\omega^3) \to \omega$ by $\pi(x \restriction n, y \restriction n, z \restriction n)$ is the position of $n-1$ in the order $R(x, y, z) \restriction \{0, \cdots, n-1\}$. Define the tree T (which will be called the tree associated with the Σ_3^1 set A) as follows: T will be a subtree of $\mathrm{Seq}(\omega \times \omega \times \alpha)$ where α is any ordinal $> j^n(k)$ for all $n < \omega$.

$$\langle x \restriction n, y \restriction n, \langle \beta_0, \cdots, \beta_{n-1} \rangle \rangle \in T$$

iff for all $0 \leqq i \leqq n-2$ let $l = \text{length}(s_i)$ and let s_k be the first $l-1$ elements of s_i (by assumption on the enumeration $\{s_0, s_1, \cdots\}$ of $\mathrm{Seq}(\omega)$, $l \leqq i$, $k < i$) then we have

$$j_{\pi(x \restriction l, y \restriction l, s_k)}(\beta_k) > \beta_i.$$

THEOREM 1.1. (M.C.) (Mansfield [13], Martin–Solovay [14]) *$x \in A$ iff T_x is not well founded.*

THEOREM 1.2. (M.C.) (Mansfield [13]) *Let $L[T]$ be the collection of all sets constructible from the tree T. Assume $L[T] \cap \omega^\omega$ is countable, then the Σ^1_3 set A is Lebesgue measurable, has the Baire property, and is either countable or contains a perfect subset.*

The reason we reproduced the definition of the tree T in so much detail is to show that it is rather absolute.

LEMMA 1.3. *Let M be any transitive class containing the parameters of the Σ^1_3 definition of A. (We do not assume necessarily that $M \subseteq V$, M can be a subclass of some generic extension of V.) Assume further that κ is still measurable in M and in M we can find a normal ultrafilter on κ, U^M, such that the iterates of the corresponding elementary embedding j^M_n satisfy $j^M_n(\alpha) = j_n(\alpha)$ for all α, then the tree associated with the Σ^1_3 set A is the same in V and in M.*

PROOF. Obvious from the definition of the tree T and the fact that the definition of R is absolute. □ Lemma 1.3

The main theorem of this section is:

THEOREM 1.4. (M.C.) *If ω_1 carries a precipitous ideal then every Σ^1_3 set is Lebesgue measurable, has the Baire property, and is either countable or contains a perfect subset.*

PROOF. The proof will simply verify that the assumption of Theorem 1.2 holds for every T which is the tree associated with some Σ^1_3 set. Fix $\mathcal{I}$ which is a precipitous ideal on ω_1, A a Σ^1_3 set, and T which is the tree associated with A. So our objective is to show that $L[T] \cap \omega^\omega$ is countable.

Consider a generic extension of V using $\mathcal{P}(\mathcal{I})$ as the set of forcing conditions and let G be the generic filter. G is an ultrafilter on $P^V(\omega_1)$ and since $\mathcal{I}$ is precipitous $\mathrm{Ult}(V, \omega_1, G)$ is well founded and isomorphic to a transitive class $M_G = M$. $i = i_G$ is the canonical embedding of V into M. Note that $\omega_1 < i(\omega_1)$ since $\mathcal{I}$ is non-principal and ω_1 complete.

LEMMA 1.5. *If α is inaccessible in V then $i(\alpha) = \alpha$.*

PROOF. Since we used $\mathcal{P}(\mathcal{I})$ as the set of forcing conditions which has cardinality $< \alpha$, α is still inaccessible in $V[G]$. Hence for every $\beta < \alpha$, $i(\beta)$ is at most the cardinal succeeding the cardinality in $V[G]$ of functions from ω^V_1 into β, which is less then α. Note also $i(\alpha) = \sup\{i(\beta) \mid \beta < \alpha\}$ since α is regular and every function from ω_1 into α is actually a function from ω_1 into β for some $\beta < \omega_1$. Hence $i(\alpha) \leqq \alpha$ therefore $i(\alpha) = \alpha$. □ Lemma 1.5

It follows from Lemma 1.5 that $i(\kappa) = \kappa$, hence κ is measurable in M and $i(U_n)$ is in M a κ complete ultrafilter on $[\kappa]^n$. Similarly $i(j_n)$ is an embedding of $\mathrm{Ult}(M, \kappa, i(U_n))$ into $\mathrm{Ult}(M, \kappa, i(U_{n+1}))$. We shall abuse the language again by denoting for an ultrafilter U in V the filter generated by it in $V[G]$ also by U.

LEMMA 1.6. $i(U) = U \cap M$.

PROOF. Let $A \in i(U)$. First assume that A is in the range of i, i.e., $A = i(B)$ for some $B \subseteq \kappa$. Since i is an elementary embedding, $B \in U$.

Let $B' = \{\alpha \mid \alpha \in B, V \models \alpha \text{ is inaccessible}\}$. Clearly $B' \in U$ since U is normal.

CLAIM. $B' \subseteq A$ because for $\alpha \in B'$, $i(\alpha) = \alpha$, hence

$$\alpha = i(\alpha) \in i(B') \subseteq i(B) = A.$$

Now consider general $A \in i(U)$. A is represented in the ultrapower $\mathrm{Ult}(V, \omega_1, G)$ by the function $f \in V$. Without loss of generality since $A \in i(U)$, and $i(U)$ is represented the constant function U, $f \in U^{\omega_1}$. U is κ complete and $\omega_1 < \kappa$, hence $B = \bigcap_{\alpha < \omega_1} f(\alpha) \in U$. Clearly $i(B) \subseteq A$ in view of Łos theorem since $B \subseteq f(\alpha)$ for all $\alpha < \omega_1$. The argument we had above shows that $i(B)$ includes a set in U, hence the same applies to A and A is in the filter generated by U. Hence $i(U) \subseteq U \cap M$.

If $A \not\in i(U)$ then $\kappa - A \in i(U)$. Apply the previous argument to $\kappa - A$ and get that A cannot include a set in U, hence $i(U) \supseteq U \cap M$. □ Lemma 1.5

LEMMA 1.7. $i(U_n) = U_n \cap M$.

PROOF. Is clear from the definition of U_n in terms of U and Lemma 1.6. □ Lemma 1.7

The following lemmata are introduced to show that $\mathrm{Ult}(\mathrm{Ord}, [\kappa]^n, U^n)$ is essentially the same as $\mathrm{Ult}^M(\mathrm{Ord}, [\kappa]^n, i(U_n))$.

LEMMA 1.8. *Let $g \in V[G]$ by a function from $[\kappa]^n$ into V, then g is equivalent modulo U_n to a function $g' \in V$.*

PROOF. This is essentially proved by Levy–Solovay [11], since $V[G]$ was obtained from V by forcing with a set of cardinalty $< \kappa$. □ Lemma 1.8

LEMMA 1.9. *Let $g \in V$ be a function from $[\kappa]^n$ into* Ord, *then there exists $g' \in M$ which is equivalent to g* modulo U_n.

PROOF. For every ordinal β, there exists a function in V, $h_\beta : \omega_1 \to \mathrm{Ord}$, which represents β in the ultrapower $\mathrm{Ult}(V, \omega_1, G)$. Of course the assignment $\beta \to h_\beta$ is not necessarily describable in V.

Consider the map (in $V[G]$) defined on $[\kappa]^n$ by $\{\alpha_1, \cdots, \alpha_n\} \to h_{g(\{\alpha_1, \cdots, \alpha_n\})}$. It is a function from $[\kappa]^n$ into V. Apply Lemma 1.8 and get $H \in V$ such that

$$B = \{\{\alpha_1, \cdots, \alpha_n\} \mid H(\{\alpha_1, \cdots, \alpha_n\}) = h_{g(\{\alpha_1, \cdots, \alpha_n\})}\} \in U_n.$$

Without loss of generality we can assume that for every $\{\alpha_1, \cdots, \alpha_n\} \in [\kappa]^n$, $H(\{\alpha_1, \cdots, \alpha_n\}) \in \mathrm{Ord}^{\omega_1}$.

We are going to define g'. In order to show that it is in M, g has to be the equivalence class in $\mathrm{Ult}(V, \omega_1, G)$ of some function from ω_1 into V, which we can assume with loss of generality get values which are functions from $[\kappa]^n$ into Ord. So define S on ω_1 by (for $\mu < \omega_1$)

$$S(\mu) = \text{The function defined on } [\kappa]^n \text{ by } \{\alpha_1, \cdots, \alpha_n\} \to H(\{\alpha_1, \cdots, \alpha_n\})(\mu).$$

The equivalence class of S modulo G is clearly a member of M which is a function from $i([\kappa]^n) = [\kappa]^n$ into Ord. Denote this function by g'. We claim that $\{\{\alpha_1, \cdots, \alpha_n\} \mid g'(\{\alpha_1, \cdots, \alpha_n\}) = g(\{\alpha_1, \cdots, \alpha_n\})\}$ includes a set in U_n. In fact this set is

$$B' = \{\{\alpha_1, \cdots, \alpha_n\} \mid \alpha_i \text{ is inaccessible (in V)} \quad \text{for } 1 \leqq i \leqq n\} \cap B,$$

$B' \in U_n$ since U is normal. Let $\{\alpha_1, \cdots, \alpha_n\} \in B'$. Note that by Lemma 1.5 $i(\{\alpha_1, \cdots, \alpha_n\}) = \{\alpha_1, \cdots, \alpha_n\}$; $\{\alpha_1, \cdots, \alpha_n\}$ is represented in $\mathrm{Ult}(V, \omega_1, G)$ by the constant function $\{\alpha_1, \cdots, \alpha_n\}$. Hence $g'(\{\alpha_1, \cdots, \alpha_n\})$ is represented by the function defined on $\mu < \omega_1$ by

$$S(\mu)(\{\alpha_1, \cdots, \alpha_n\}) = H(\{\alpha_1, \cdots, \alpha_n\})(\mu).$$

Since $\{\alpha_1, \cdots, \alpha_n\} \in B$, $H(\{\alpha_1, \cdots, \alpha_n\})$ is $h_{g(\{\alpha_1, \cdots, \alpha_n\})}$ hence $g'(\{\alpha_1, \cdots, \alpha_n\})$ is represented by the function $h_{g(\{\alpha_1, \cdots, \alpha_n\})}$. By definition $[h_{g(\{\alpha_1, \cdots, \alpha_n\})}]_G = g(\{\alpha_1, \cdots, \alpha_n\})$, hence we proved that for $\{\alpha_1, \cdots, \alpha_n\} \in B'$, $g(\{\alpha_1, \cdots, \alpha_n\}) = g'(\{\alpha_1, \cdots, \alpha_n\})$ and $g \equiv_{U_n} g'$. □ Lemma 1.9

LEMMA 1.10. *Let $g \in V \cap \mathrm{Ord}^{[\kappa]^n}$. Let g' be the function which is equivalent to* g modulo U_n, *which is guaranteed to exist by Lemma* 1.9. *Then* $[g]_{U_n} = [g']_{i(U_n)}$. *(The meaning of this equality is that the ordinal represented in* $\mathrm{Ult}(V, [\kappa]^n, U_n)$ *by* g *is the same as the ordinal represented by* g' *in* $\mathrm{Ult}(M, [\kappa]^n, i(U_n))$.)

PROOF. $[g]_{U_n}$ is of course an ordinal and we prove the lemma by induction on $[g]_{U_n}$. If $[g]_{U_n} = \alpha$ pick, for every $\beta < \alpha$, g_β such that $[g_\beta]_{U_n} = \beta$. By induction assumption $[g'_\beta]_{i(U_n)} = \beta$, where $g'_\beta \in M$ is equivalent modulo U_n to g_β. Hence for $\beta < \alpha$, $\{\{\alpha_1, \cdots, \alpha_n\} \mid g'_\alpha(\{\alpha_1, \cdots, \alpha_n\}) < g'(\{\alpha_1, \cdots, \alpha_n\})\}$ includes a set in U_n. By Lemma 1.6 it implies that for $\beta < \alpha$, $[g'_\beta]_{i(U_n)} < [g']_{i(U_n)}$. Hence $\alpha \leqq [g']$.

Assume $\alpha < [g']_{i(U_n)}$, therefore there exists a function in M, h, such that $[g'_\beta]_{i(U_n)} < [h]_{i(U_n)} < [g']_{i(U_n)}$ for all $\beta < \alpha$. By Lemma 1.8, h is equivalent modulo U_n to a function $\tilde{h}$ in V, but then by $g'_\beta \equiv_{U_n} g_\beta$, $g' \equiv_{U_n} g$, $h \equiv_{U_n} \tilde{h}$ and by Lemma 1.6 we get $\beta = [g_\beta]_{U_n} < [\tilde{h}]_{U_n} < [g]_{U_n}$ for all $\beta < \alpha$. Hence $\alpha \leqq [\tilde{h}]_{U_n} < [g]_{U_n}$, which is a contradiction. □ Lemma 1.10

$i(B)$ for a class B is defined as $i(B) = \bigcup_{x \in V} i(B \cap x)$.

Lemma 1.11. $i(j_n) \restriction \mathrm{Ord} = j_n \restriction \mathrm{Ord}$.

Proof. Let $\alpha \in \mathrm{Ord}$. Let $g \in V \cap \mathrm{Ord}^{[\kappa]^n}$ be such that $[g]_{U_n} = \alpha$. Define $h \in V \cap \mathrm{Ord}^{[\kappa]^{n+1}}$ by $h(\{\alpha_1, \cdots, \alpha_{n+1}\}) = g(\{\alpha_1, \cdots, \alpha_n\})$. $j_n(\alpha)$ is simply $[h]_{U_{n+1}}$. Let us find what is $i(j_n)$ (α). By Lemma 1.10, $\alpha = [g']_{i(U_n)}$ where $g' \in M$ is equivalent to g modulo U_n. (Use Lemma 1.9.) Define h' in M by (h' is defined on $[\kappa]^{n+1}$)

$$h'(\{\alpha_1, \cdots, \alpha_{n+1}\}) = g'(\{\alpha_1, \cdots, \alpha_n\}).$$

Since $[g']_{U_n} = \alpha$, $i(j_n)(\alpha)$ is $[h']_{i(U_{n+1})}$. Clearly h' is equivalent to h modulo U_{n+1}. (Thus the liberty we took in denoting it by h' is justified.) But then by Lemma 1.10, $[h']_{i(U_{n+1})} = [h]_{U_{n+1}}$, hence $j_n(\alpha) = i(j_n)(\alpha)$. □ Lemma 1.11

Corollary 1.12. *$i(T) = T$, hence the tree T is in M, hence $L^M[T] = L[T]$.*

Proof. Simply use the same definition of T in M and in V. By Lemma 1.11 the assumptions of Lemma 1.3 are satisfied, hence this definition in M gives the same tree T, therefore $i(T) = T$. (Note that i is the identity on $V \cap \omega^\omega$, hence any parameter in the Σ^1_3 definition of A is a fixed point of i. Similarly $i(R) = R$ since R is considered to be a function from finite sequences of ω into ω.)
□ Corollary 1.12

Lemma 1.13. *In V, $L[T] \cap \omega^\omega$ is countable.*

Proof. Assume otherwise. Hence we have a sequence $\langle a_\mu \mid \mu < \omega_1 \rangle$ of different members of $L[T] \cap \omega^\omega$. The function $g(\mu) = a_\mu$ has an equivalence class which in $\mathrm{Ult}(V, \omega_1, G)$ represents a member of $L^M[i(T)] \cap \omega^\omega$ by Łos theorem since this statement is true for every single value of this function. By Corollary 1.12, $i(T) = T$, $L^M[(T)] = L[T]$, hence $[g]_G \in V \cap L[T] \cap \omega^\omega$. Therefore there exist x in V such that $[g]_G = x$, but $x \in \omega^\omega$ hence $i(x) = x$.

Since $i(x)$ is represented in $\mathrm{Ult}(V, \omega_1, G)$ by the constant function x, $\{\mu \mid a_\mu = x\} \in G$, but this is an impossibility since $|\{\mu \mid a_\mu = x\}| \leqq 1$ (remember $\langle a_\mu \mid \mu < \omega_1 \rangle$ is a sequence of different reals) and $G \subseteq \mathcal{P}(\mathcal{I})$. □ Lemma 1.13

Proof of Theorem 1.14 is now immediate from Lemma 1.13 and Theorem 1.2.

□ Theorem 1.14

LEMMA 1.14. *Lemmata* 1.6–1.12 *hold if we replace $\mathcal{I}$ by any precipitous ideal on a set I, G by a generic filter in $\mathcal{P}(\mathcal{I})$, M by* $\mathrm{Ult}(V, I, G)$ *and i by the canonical embedding of V into M provided $|I| < \kappa$.*

PROOF. The only fact about ω_1 we used in the above lemmata is $\omega_1 < \kappa$.

□ Lemma 1.14

2. Σ_4^1 are Lebesgue measurable and has the Baire property

In this section we make much stronger assumptions on the existence of precipitous ideals and get the same type of result we were able to get for Σ_3^1 sets to apply to Σ_4^1 sets. Before we state the main theorem of this section we need one basic fact:

THEOREM 2.1. *Let $\mathcal{I}$ be a precipitous ideal on a given set I. $V[G]$ is a generic extension of V using a set of conditions which satisfies the countable chain condition (c.c.c.). Then in $V[G]$ $\mathcal{I}$ generates a precipitous ideal. If I is $P_{\omega_1}(\lambda)$ and the ideal $\mathcal{I}$ was normal, it generates a normal ideal.*

PROOF. The ideal generated by $\mathcal{I}$, $\mathcal{I}^*$, is the set of all sets in $V[G]$ which are included in some set in $\mathcal{I}$. Note that if τ is a forcing term which is forced to be in $\mathcal{I}^*$ then we can find $A \in \mathcal{I}$ such that it is forced that $\tau \subseteq A$. (By c.c.c. we get a countable collection of members of $\mathcal{I}$ such that it is forced that τ is included in one of them, but then their union, which is still in $\mathcal{I}$ by the ω_1 completeness of $\mathcal{I}$, is forced to include τ.)

It follows that $\mathcal{I}^*$ is ω_1 complete because if $\langle \tau_n \mid n < \omega \rangle$ is forced to be a sequence of members of $\mathcal{I}^*$, find the sequence $\langle A_n \mid n < \omega \rangle$ such that it is forced that $\tau_n \subseteq A_n$ and then $\bigcup_{n<\omega} \tau_n$ is forced to be included in $\bigcup_{n<\omega} A_n$ which is in $\mathcal{I}$, hence $\bigcup_{n<\omega} \tau_n$ is in $\mathcal{I}^*$.

Consider $\mathcal{F} \in V$ which is a family of mutually incompatible elements of $\mathcal{P}(\mathcal{I})$ (namely the elements of $\mathcal{F}$ are subsets of I, not in $\mathcal{I}$, such that the intersection of any two is in $\mathcal{I}$) and $\mathcal{F}$ is maximal with this property.

LEMMA 2.2. *Let $\mathcal{F} \in V$ be as above, then in $V[G]$ $\mathcal{F}$ is a maximal family of mutually incompatible elements of $\mathcal{P}(\mathcal{I}^*)$.*

PROOF. Clearly $\mathcal{F}$ is a family of members of $\mathcal{P}(\mathcal{I})$ and the intersection of any two is in $\mathcal{I}^*$. Assume $\mathcal{F}$ is not maximal. Let τ be a term denoting a subset of I which is forced by a condition p to be in $\mathcal{P}(\mathcal{I})-\mathcal{I}^*$ and such that the intersection of it with every member of $\mathcal{F}$ is in $\mathcal{I}^*$.

For every $A \in \mathcal{F}$ we can find (in V) a set $B(A) \in \mathcal{I}$ such that $p \Vdash \tau \cap A \subseteq B(A)$ (argument as before). Let

$$D = \{i \mid i \in I, \text{some } q \text{ which extends } p \text{ forces } i \in \tau\}.$$

Clearly $D \notin \mathcal{I}$ (otherwise $p \Vdash \tau \subseteq D$ and $p \Vdash \tau \in \mathcal{I}^*$). Since $\mathcal{F}$ is a maximal family of mutually incompatible members of $\mathcal{P}(\mathcal{I})$, $D \in \mathcal{P}(\mathcal{I})$ hence there is $A \in \mathcal{F}$ such that $A \cap D \notin \mathcal{I}$. But then $A \cap D - B(A) \notin \mathcal{I}$ since $B(A) \in \mathcal{I}$. Pick $i \in A \cap D - B(A)$. Since $i \in D$ we have q which extends p and $q \Vdash i \in \tau$. Hence $q \Vdash i \in \tau \cap A - B(A)$ which is a contradiction to $p \Vdash \tau \cap A \subseteq B(A)$.

□ Lemma 2.2

COROLLARY 2.3. *If U is generic in $\mathcal{P}(\mathcal{I}^*)$ over $V[G]$ then $U \cap \mathcal{P}(\mathcal{I})$ is generic in $\mathcal{P}(\mathcal{I})$ over V.*

LEMMA 2.4. *Let U be generic in $\mathcal{P}(\mathcal{I}^*)$ over $V[G]$ and let g be a function $g \in V[G]$, $g: I \to \mathrm{Ord}$, then there exists $h \in V$ such that $\{i \mid g(i) = h(i)\} \in U$.*

PROOF. Assume otherwise. This is a statement about $V[G][U]$, hence it is forced over $V[G]$ by some member of $\mathcal{P}(\mathcal{I}^*)$. This member of $\mathcal{P}(\mathcal{I}^*)$ is the realization with respect to G of some term τ. So the fact that τ forces over $V[G]$ this fact about g is forced over V by some $p \in G$, i.e., let μ be a term whose realization is g:

(I) $\quad p \Vdash$ "$\tau \Vdash_{\mathcal{P}(\mathcal{I}^*)}$ no function in V is equivalent modulo U to μ".

As in Lemma 2.2 let

$$D = \{i \mid \text{Some } q \text{ which extends } p \text{ forces } i \in \tau\}.$$

Clearly $D \notin \mathcal{I}$.

For each $i \in D$ pick q_i which extends p and which forces $i \in \tau$ and also $\mu(i) = \alpha$ for some ordinal α.

Define h on I (note that this definition takes place in V):

$$h(i) = \begin{cases} \text{the value forced by } q_i \text{ for } \mu(i) & \textit{if } i \in D, \\ 0 & \textit{otherwise.} \end{cases}$$

CLAIM. There exists an extension of p, q such that

$$q \Vdash \{i \mid i \in \tau, q_i \in G\} \notin \mathcal{I}^*.$$

PROOF OF THE CLAIM. Otherwise $p \Vdash \{i \mid i \in \tau, q_i \in G\} \in \mathcal{I}^*$. Hence we can find $B \in \mathcal{I}$ such that

$$\text{(II)} \qquad p \Vdash \{i \mid i \in \tau, q_i \in G\} \subseteq B.$$

Pick $i \in D - B$, q_i extends p and it forces $i \in \tau$ and $q_i \in G$. This is clearly a contradiction to (II), since q_i extends p, and the claim is verified.

Let q be as in the claim. Let ρ be the term denoting the set $\{i \mid i \in \tau, q_i \in G\}$. By the definition q

$$q \Vdash \rho \notin \mathcal{I}^* \wedge \rho \subseteq \tau.$$

Hence q forces that ρ extends τ as elements of $\mathcal{P}(\mathcal{I}^*)$. Consider now the definition of h. It should be evident that $\rho \subseteq \{i \mid \mu(i) = h(i)\}$. Hence over $V[G]$ ρ forces in $\mathcal{P}(\mathcal{I}^*)$ that the function g, which is the realization of μ, is equivalent modulo U to h. Hence

$q \Vdash$ "τ does not force that μ is not equivalent modulo U to a member of V",

since q forces that some extensions of τ in $\mathcal{P}(\mathcal{I}^*)$ do force $g \equiv_U h$ and $h \in V$. This obviously contradicts (I) and "q extends p". □ Lemma 2.4

PROOF OF THEOREM 2.1. Now let U be generic in $\mathcal{P}(\mathcal{I}^*)$ and assume that $\mathrm{Ult}(V[G], I, U)$ is not well founded. (This fact is, by Lemma 0.2, equivalent to $\mathcal{I}^*$ not precipitous.) It means that there is a sequence of functions $\langle g_n \mid n < \omega \rangle$, $g_n \in V[G]$, $g_n : I \to \mathrm{Ord}$ and for each $n < \omega$, $\{i \mid g_{n+1}(i) < g_n(i)\} \in U$. (We are not assuming that the sequence $\langle g_n \mid n < \omega \rangle$ is in $V[G]$.)

Apply Lemma 2.4 and get for each g_n, h_n in V such that $h_n \equiv_U g_n$. But then clearly for all n

$$\{i \mid h_{n+1}(i) < h_n(i)\} \in U \cap V.$$

By Corollary 2.3, $U \cap V$ is generic in $\mathcal{P}(\mathcal{I})$ over V. Hence $\mathrm{Ult}(V, I, U \cap V)$ should be well founded since $\mathcal{I}$ is precipitous. The sequence $\langle h_n \mid n < \omega \rangle$ is an obvious counterexample, hence we get a contradiction and we have proved that $\mathcal{I}^*$ is precipitous.

We proved Theorem 2.1 except for the case $I = P_{\omega_1}(\lambda)$ and $\mathcal{I}$ normal in which we have to show that $\mathcal{I}^*$ is still normal. The first clause of normality, i.e. for $\alpha < \lambda$, $\{P \mid \alpha \notin P,\ P \in P_{\omega_1}(\lambda)\} \in \mathcal{I}^*$ follows immediately from the corresponding fact for $\mathcal{I}$. For the second clause assume that τ is forced to be a choice function on a subset of $P_{\omega_1}(\lambda)$, μ, which is in $\mathcal{P}(\mathcal{I}^*)$. Let p force these facts.

Assume that τ is not constant on any subset of μ, which is in $\mathcal{P}(\mathcal{J}^*)$. Hence for every $\alpha \in \lambda$ the set

$$\mu_\alpha = \{P \mid P \in \mu, \tau(P) = \alpha\} \in \mathcal{J}^*.$$

Hence we can find $A_\alpha \in \mathcal{J}$ such that $p \Vdash \mu_\alpha \subseteq A_\alpha$.

CLAIM. $B = \{P \mid P \in \bigcup_{\alpha \in P} A_\alpha\} \in \mathcal{J}$ (standard argument). Otherwise for each $P \in B$ pick $\alpha \in P$ such that $P \notin A_\alpha$. This is a choice function which is not constant on any member of $\mathcal{P}(\mathcal{J})$.

Clearly $p \Vdash \mu \subseteq B$ (for $P \in \mu$, $\tau(P) \in P$, hence $P \in \mu_{\tau(P)} \subseteq A_{\tau(P)}$, which implies $P \in B$) hence μ is in $\mathcal{J}^*$ and we get a contradiction. □ Theorem 2.1

THEOREM 2.5. (M.C.) *Let $\mathcal{P}$ be a forcing notion of cardinality λ where $\lambda < \kappa$ for some measurable cardinal κ. Let $\lambda \leqq \mu$ be the cardinality of all dense subset of $\mathcal{P}$ and let $\mathcal{J}$ be a normal precipitous ideal on $P_{\omega_1}(\mu)$. Then if G is $\mathcal{P}$ generic over V, Σ_4^1 statements are absolute between V and $V[G]$.*

PROOF. It follows from Mansfield [13] that Π_3^1 statements are absolute for Cohen extensions using a set of forcing conditions of cardinality $< \kappa$. Hence if we are given a Σ_4^1 statement $\exists z \Phi(z)$ (Φ is Π_3^1 with possible parameters) which holds in V for some z as witness, we get $V[G] \models \Phi(z)$. Hence $V[G] \models \exists z \Phi(z)$.

The converse direction, i.e., $V[G] \models \exists z \Phi(z)$ implies $V \models \exists z \Phi(z)$, is more interesting. So assume $V[G] \models \exists z \Phi(z)$. Let τ be a term in the forcing language for $\mathcal{P}$ which is forced by $p \in \mathcal{P}$ to satisfy $\Phi(\tau)$. Without loss of generality we may assume that p is the minimal element in $\mathcal{P}$. Let T be the tree associated (as in section 1) with the Σ_3^1 set $A = \{z \mid \neg \Phi(z)\}$. Since $V[G]$ was obtained from V by forcing with a set of forcing conditions of cardinality $< \kappa$ it follows from Mansfield [13] that T is still the tree associated with A in $V[G]$. Hence

(III) $\qquad p \Vdash T_\tau$ is well founded.

Now force (over V) with $\mathcal{P}(\mathcal{J})$ where $\mathcal{J}$ is the given precipitous normal ideal over $P_{\omega_1}(\mu)$. U is the generic filter, hence $\mathrm{Ult}(V, P_{\omega_1}(\mu), U)$ is well founded and its transitive isomorph is M. i again is the natural embedding of V into M, and by Lemma 1.14, $i(T) = T$.

For $\alpha \leqq \mu$ let $f_\alpha : P_{\omega_1}(\mu) \to \omega$ be defined by $f_\alpha(P) =$ the order type of $P \cap \alpha$.

LEMMA 2.6. *For $\alpha \leqq \mu$ and for every generic $U \subseteq \mathcal{P}(\mathcal{J})$, $[f_\alpha]_U = \alpha$.*

PROOF. By induction on α. Using the normality of $\mathcal{J}$ we get that for U generic any choice function on $P_{\omega_1}(\mu)$ which is in V is almost constant with respect to U. Hence if $[g]_U < [f_\alpha]_U$, define the choice function

$$\tilde{g}(P) = \text{The } g(P)\text{th element of } P \text{ if } g(P) < f_\alpha(P) \qquad \text{and}$$

$$\tilde{g}(P) = \text{The minimal element of } P \text{ otherwise.}$$

Clearly $\tilde{g}$ is modulo U the constant β for $\beta < \alpha$ hence $[g] \equiv_U [g_\beta] = \beta$ by induction assumption. Thus we have proved that

$$[f_\alpha]_U = \sup\{[g]_U + 1 \mid [g]_U < [f_\alpha]_U\} \leqq \sup\{\beta + 1 \mid \beta < \alpha\} = \alpha.$$

The converse inequality $[f_\alpha]_U \geqq \alpha$ follows easily from $[f_\alpha]_U > [f_\beta]_U$ since $\{P \mid \beta \notin P, P \in P_{\omega_1}(\mu)\} \in \mathcal{I}$. □ Lemma 2.6

COROLLARY 2.7. *$M \models \mu$ is countable.*

PROOF. By Lemma 2.6 the function representing μ is f_μ which is a countable ordinal everywhere, hence by Łos theorem, μ is countable. □ Corollary 2.7

LEMMA 2.8. *Let $B \in V$, $B \subseteq \mu$ then $B \in M$.*

PROOF. Define h on $P_{\omega_1}(\mu)$ by

$$h(P) = \{f_\alpha(P) \mid \alpha \in B \cap P\}.$$

We claim that $[h]_U = B$. By normality of $\mathcal{I}$ we have that for all $\alpha < \mu$, $\{P \mid \alpha \in P, P \in P_{\omega_1}(\mu)\} \in U$, hence if $\alpha \in B$, $[f_\alpha]_U \in [h]_U$, which by Lemma 2.6 implies $\alpha \in [h]_U$. Therefore $B \subseteq [h]_U$.

For the other direction let $[g]_U \in [h]_U$. Without loss of generality we may assume that for every P, $g(P) = f_{\alpha(P)}(P)$ for some $\alpha(P) \in B \cap P$. $\alpha(P)$ is a choice function on $P_{\omega_1}(\mu)$, hence by normality of $\mathcal{I}$ it is constant on a set in U. Let α be this constant. Hence $\{P \mid g(P) = f_\alpha(P)\} \in U$ and obviously $\alpha \in B$. Thus for some $a \in B$, $[g]_U = [f_\alpha]_U = \alpha$ (By Lemma 2.6). Thus we have shown $[h]_U \subseteq B$.
□ Lemma 2.8

Without loss of generality we may assume that $\mathcal{P}$ is a partial order defined on $\lambda = |\mathcal{P}|$. So as a consequence of Lemma 2.8 we get

LEMMA 2.9. *M is as before.*

(a) *The set of all dense subsets of P which are in V, is in M and is countable there.*

(b) *In M we can find a $\mathcal{P}$ generic filter over V, G.*

(c) *If $G \in M$ and G is $\mathcal{P}$ generic over V, then every real of $V[G]$ is in M.*

PROOF. (a) Since the cardinality of dense subsets of $\mathcal{P}$ is μ, we can code (in

V) on enumeration in order type μ of all this dense subsets by $B \subseteq \mu$. By Lemma 2.8, $B \in M$, hence since we may assume that the coding is absolute the set of all subsets of $\mathcal{P}$ which are dense and in V is in M. It is countable there (since μ is countable in M by Corollary 2.7).

(b) Follows from (a) by picking an enumeration of order type ω of all dense subsets of $\mathcal{P}$ which lie in V and defining the generic filter by induction.

(c) Every real, x, in $V[G]$ is the realization of some term in V which is forced to denote a real. Such a term can be considered to be a double sequence $\langle E_{n,m} \mid n, m < \omega \rangle$ where $E_{n,m} \subseteq \mathcal{P}$. ($E_{n,m}$ is the set of elements of $\mathcal{P}$ forcing $x(n) = m$.) This sequence can easily be coded as a subset of $\lambda = |\mathcal{P}|$. Hence it is in M. Since G is in M, x is in M by the definition $x(n) = m$ if and only if $E_{n,m} \cap G \neq \emptyset$. □ Lemma 2.9

Conclusion of the Proof of Theorem 2.5. By Lemma 2.9 fix $G \in M$ which is $\mathcal{P}$ generic over V. Note again that by Lemma 2.9, z, the realization of the term τ with respect to G is in M. By (III)

$$V[G] \models T_z \text{ is well founded.}$$

Since the tree T and z are in M

$$M \models T_z \text{ is well founded.}$$

Note that $i(T) = T$, therefore T is in M, the tree associated with $A = \{z \mid \neg\Phi(z)\}$. Therefore $M \models \Phi(z)$, hence $M \models \exists z \Phi(z)$. Since i is an elementary embedding of V into M and it is the identity on any parameter of Φ we get $V \models \exists z \Phi(z)$, which is exactly that we were trying to prove. □ Theorem 2.5

Corollary 2.10. *Let μ be a supercompact cardinal which has unboundedly many measurable cardinals above it. In the model obtained by the standard collapse of μ to ω_1, Σ_4^1 statements are preserved under generic extensions.*

Proof. Arguments of [6] show that in the resulting model $P_{\omega_1}(\lambda)$ carries a precipitous normal ideal for every λ. Now use Theorem 2.5. □ Corollary 2.10

Theorem 2.11. (M.C. + C.H.) *Assume ω_1 carries a precipitous ideal and $P_{\omega_1}(2^{2^{\omega_1}})$ carries a normal precipitous ideal, then every Σ_4^1 set is Lebesgue measurable and has the Baire property.*

Proof. The proof relies very heavily on ideas of Solovay from [18] and [19].

Recall that a real x is called random (respectively generic) over V if x does not lie in any Borel set whose description in some standard way is in V and which has measure 0 (respectively, is of first category). An equivalent definition is that the collection of Borel sets with description in V such that x is a member of them is a generic filter over V, in the forcing notion of Borel sets modulo sets of measure 0 (respectively, of first category). Note that both forcing notions satisfy the countable chain condition.

Let $\mathcal{I}$ be a precipitous ideal on ω_1 and U a $\mathcal{P}(\mathcal{I})$ generic filter over V. M is the transitive isomorph of $\mathrm{Ult}(V, \omega_1, U)$. i is the natural embedding of V into M. Note that $i(\omega_1) > \omega_1$, hence ω_1 is countable in M. Note also that every $A \subseteq \omega_1$, $A \in V$ is in M (since $i(\alpha) = \alpha$ for countable $A = i(A) \cap \omega_1$ which is clearly in M).

The following lemma is standard (see [18], [19]).

Lemma 2.12. (a) *The reals in M which are random over V are of measure* 1 *in M.*

(b) *The reals in M which are generic over V are a set whose complement is of first category.*

Proof. (a) ω_1^V is countable in M. We have ω_1^V many Borel sets in V (here we use the continuum hypothesis) and each of them can be coded by a real in V. Hence we can code a list of all Borel sets in V which are of measure 0 by a subset of ω_1. Hence this list is in M. Each element of the list is of measure 0 also in M (since its code is invariant under i), hence the union of the element of the list is in M a countable union of Borel sets of measure 0, hence of measure 0. Any real not in the union is random over V, therefore the set of random reals has measure 1.

(b) Like (a) replace everywhere "measure 0" by "first category".

□ Lemma 2.12

Lemma 2.13. *Let $a \in M$. a is either random or generic over V. Let Φ be a Π^1_3 formula (with possible parameters from V). Then*

$$V[a] \models \exists z\, \Phi(z, a) \quad \textit{iff}\ M \models \exists z\, \Phi(z, a).$$

Proof. Note that every real in $V[a]$ is the realization of some term in the forcing language for forcing with Borel sets modulo either sets of measure 0 or sets of first category. Such a term can be easily coded by a real in V, hence it is in M. Since a is in M, we get that the realization of this term by the generic filter generated by a is in M. Thus we have proved that every real in $V[a]$ is in M.

Let T be the tree associated in V with the Σ^1_3 set $A = \{x * y \mid \neg\ \Phi(x, y)\}$. Since $V[a]$ and $V[U]$ were obtained from V by forcing with a forcing notion of cardinality less than μ, T is still the tree associated with A in $V[a]$ and in $V[U]$, and in view of Lemma 1.14 it is the tree associated with A in M.

Assume $V[a] \models \exists z \Phi(z, a)$. Fix such z. By definition T

$$V[a] \models T_{z*a} \text{ is well founded.}$$

Since z, a are in M

$$M \models T_{z*a} \text{ is well founded.}$$

Hence $M \models \Phi(z, a)$, therefore $M \models \exists z \Phi(z, a)$.

For the converse implication assume $M \models \exists z \Phi(z, a)$. Fix such z. Hence

$$M \models T_{z*a} \text{ is well founded.}$$

Since $M \subseteq V[U]$

$$V[U] \models T_{z*a} \text{ is well founded.}$$

Therefore

$$V[U] \models \Phi(z, a) \text{ which implies } V[U] \models \exists z \Phi(z, a),$$

$V \subseteq V[a] \subseteq V[U]$ and by well known results (see Jech [4]) $V[U]$ is a generic extension of $V[a]$. (Remember that $V[U]$ is a generic extension of V using $\mathcal{P}(\mathcal{I})$ as the set of forcing conditions.) The set of forcing conditions by which $V[U]$ is obtained from $V[a]$ is a subset of $\mathcal{P}(\mathcal{I})$, $\mathcal{Q}$, and moreover any dense subset of $\mathcal{Q}$ in $V[a]$ is the intersection of some dense subsets of $\mathcal{P}(\mathcal{I})$ in V with $\mathcal{Q}$. $\mathcal{P}(\mathcal{I})$ has cardinality $\leqq 2^{\omega_1}$, hence the cardinality of its dense subsets in V is at most $\lambda = 2^{2^{\omega_1}}$. Hence the same applies to $\mathcal{Q}$ in $V[U]$.

In V, $P_{\omega_1}(\lambda)$ carried a normal precipitous ideal. By Theorem 2.1 in $V[a]$, $P_{\omega_1}(\lambda)$ still carries a normal precipitous ideal since $V[a]$ is obtained from V by a c.c.c. generic extension. Now use Theorem 2.5 in $V[a]$ and $\mathcal{Q}$ as the set of forcing conditions and get that Σ_4^1 statements are absolute between $V[a]$ and $V[U]$. Hence by (IV), $V[a] \models \exists z \Phi(z, a)$. □ Lemma 2.13

Conclusion of the Proof of Theorem 2.11. Fix a Π_3^1 formula Φ with possible parameters in V. We use arguments of Solovay from [19]. For a real random or generic over V, a, we can find a Borel set in V, B, such that

$$V[a] \models \exists z \Phi(z, a) \quad \text{iff } a \in B.$$

(It can even be the same set B for random and generic reals noting that there exists a measure 1 set of first category.) If $a \in M$ we know by Lemma 2.13 that

$$V[a] \models \exists z \Phi(z, a) \quad \text{iff } M \models \exists z \Phi(z, a).$$

Hence for a in M which is either generic or random over V

$$M \models \exists z \Phi(a, z) \quad \text{iff } a \in B.$$

By Lemma 2.12 the set of reals, random over V, is of measure 1, and the set of generic reals is a set whose complement is of first category. Therefore

(V) $\quad M \models \{x \mid \exists z \Phi(z, x)\} \Delta B$ has measure 0 and is of first category.

By definition of Lebesgue measurability and the Baire property (V) implies

$M \models \{x \mid \exists z \Phi(z, x)\}$ is Lebesgue measurable and has the Baire property.

i is an elementary embedding of V into M, which is the identity on all parameters in Φ, hence

$V \models \{x \mid \exists z \Phi(z, x)\}$ is Lebesgue measurable and has the Baire property

which is exactly the statement of Theorem 2.11. □ Theorem 2.11

A slight variation in the proof of Theorem 2.11 can give us a similar theorem. But first a definition. An ideal $\mathcal{I}$ is λ saturated if $\mathcal{P}(\mathcal{I})$ satisfies the λ chain condition. A κ^+ saturated, κ complete ideal is precipitious (see Kunen [8]) but not the other way. In fact the assumption of the existence of a ω_2 saturated ideal on ω_1 is consistencywise a much stronger assumption that the existence of precipitious ideal ([8], see also [9]).

THEOREM 2.14. (M.C. + C.H.) *If ω_1 carries an ω_1 complete ω_2 saturated ideal and $P_{\omega_1}(\omega_2)$ carries a normal precipitous ideal then every Σ^1_4 set of reals is Lebesgue measurable and has the Baire property.*

PROOF. The proof is almost a copy of the proof of Theorem 2.11. The only difference is that here we have a normal precipitous ideal on $P_{\omega_1}(\omega_2)$ rather than on $P_{\omega_1}(2^{2^{\omega_1}})$. The only place we needed it is to apply Theorem 2.5 for a forcing notion which is a subset of $\mathcal{P}(\mathcal{I})$ where $\mathcal{I}$ was precipitous on ω_1. The following lemma makes up for this difference.

LEMMA 2.15. (C.H.) *If $\mathcal{I}$ is a ω_1 complete ω_2 saturated ideal on ω_1 then $\mathcal{P}(\mathcal{I})$ has at most ω_2 dense subsets.*

PROOF. If ω_1 carries a ω_1 complete ω_2 saturated ideal and $2^\omega = \omega_1$ then $2^{\omega_1} = \omega_2$. (see [5].) Every dense subset of $\mathcal{P}(\mathcal{I})$ is determined by a maximal family of incompatible elements of $\mathcal{P}(\mathcal{I})$.

By ω_2 saturation of $\mathcal{I}$ such a family has cardinality $\leqq \omega_1$. Hence the cardinality of possible such families is at most $(2^{\omega_1})^{\omega_1} = 2^{\omega_1} = \omega_2$.

□ Lemma 2.15 □ Theorem 2.14

3. Uncountable Σ_4^1 set contains a perfect subset

The main theorem of this section is

THEOREM 3.1. (M.C.) *Assume ω_1 carries a precipitous ideal and $P_{\omega_1}(2^{2^{\omega_1}})$ carries a precipitous normal ideal then every Σ_4^1 set of reals is either countable or it contains a perfect subset.*

PROOF. Let the Σ_4^1 set be $A = \{x \mid \exists z \Phi(z, x)\}$ where Φ is Π_3^1 formula with possible parameters. Let $\mathcal{I}$ be a precipitous ideal on ω_1 and let T be the tree associated with the Σ_3^1 set $\{z * x \mid \neg \Phi(z, x)\}$. Assume further that A is uncountable.

Let U be a generic ultrafilter in $\mathcal{P}(\mathcal{I})$ and let M be the transitive isomorph of $\mathrm{Ult}(V, \omega_1, U)$. As in Lemma 1.13 pick a one to one function $g: \omega_1 \to A$. $[g]_U$ is a real in M, x, which is in the Σ_4^1 set $i(A)$. Note that $x \not\in V$ otherwise $[g]_U = x = i(x)$ and g is equivalent modulo U, to the constant function x which contradicts the fact that g is one to one. By definition of the tree T and by Lemma 1.12, $i(T) = T$,

$$M \models \exists z (T_{z*x} \text{ is well founded}).$$

$V[U]$ is obtained from U by forcing with a set of cardinality $< \kappa$. Hence in $V[U]$ (since the tree T is still the tree associated with $\{z * x \mid \neg \Phi(z, x)\}$ in $V[U]$, see [13]) we have

$$V[U] \models \exists z \Phi(z, x).$$

Now the theorem follows from Theorem 3.2 below noting as in Lemma 2.13 that the cardinality of dense subsets of $\mathcal{P}(\mathcal{I})$ is at most $2^{2^{\omega_1}}$. □ Theorem 3.1

THEOREM 3.2. (M.C.) *Let $\mathcal{P}$ be a forcing notion such that the cardinality of the family of its dense subsets is $\mu < \kappa$. Let A be a Σ_4^1 set such that forcing with $\mathcal{P}$ introduces a new member of A. Then if $P_{\omega_1}(\mu)$ carries a precipitous normal ideal, A contains a perfect subset.*

PROOF. Let $\mathcal{I}$ be a normal precipitous ideal on $P_{\omega_1}(\mu)$ and let U be a generic filter in $\mathcal{P}(\mathcal{I})$. Let M be the transitive isomorph of $\mathrm{Ult}(V, P_{\omega_1}(\mu), U)$ and let i be the natural embedding of V into M. Let $A = \{x \mid \exists z \Phi(x, z)\}$ and T the tree associated with $\{x * z \mid \neg\Phi(z, x)\}$ $(i(T) = T)$. Assume that τ is a term forced to be in A but not in V.

By Lemma 2.9 we have in M an enumeration $\langle D_n \mid n < \omega \rangle$ of all dense subsets of $\mathcal{P}$ which are in V. Moreover by part (c) of the same lemma if $G \in M$ is $\mathcal{P}$ generic over V all the reals of $V[G]$ are in M.

We now follow Solovay [19]. In M we define a tree S. Its elements will be indexed by Seq $(\{0, 1\})$ (the finite sequences of 0's and 1's). Each node indexed by s will contain an element of $\mathcal{P}, p_s$, and a natural number m_s such that p_s determines $\tau \restriction m_s$. (Remember that τ is forced to be a real, hence $\tau \restriction m_s$ is a finite sequence of members of ω.) If s extends t we assume $m_s > m_t$. The definition is by induction on $l(s)$. $m_\varnothing = 0$; p_0 is any member of D_0. Given p_s, m_s, remember that τ is forced to be outside of V, hence we can extend p_s into two conditions $p_{s\frown\{0\}}$ and $p_{s\frown\{1\}}$ and find $m_{s\frown\{0\}} = m_{s\frown\{1\}} > m_s$ such that $p_{s\frown\{0\}}$ and $p_{s\frown\{1\}}$ force contradictory information about $\tau \restriction m_{s\frown\{0\}}$. Without loss of generality we can assume that $p_{s\frown\{i\}} \in D_{l(s)+1}$ for $i \in \{0, 1\}$.

Every branch in S is determined by a member of $y \in 2^\omega$. For every such y, $\{p_{y\restriction n} \mid n < \omega\}$ clearly generates a $\mathcal{P}$ generic filter over V which we denote by G_y. For $y \in 2^\omega$ let $x(y)$ be the realization of τ with respect to G_y. The function defined on 2^ω by $y \to x(y)$ is clearly a one to one continuous mapping of 2^ω into ω^ω. Hence its range $B = \{x(y) \mid y \in 2^\omega\}$ is a perfect subset of ω^ω.

LEMMA 3.3. $M \models B \subseteq A$.

PROOF. Fix $y \in M \cap 2^\omega$. We have to show that $x(y)$ is in A. Consider G_y. By definition of τ (remember that $x(y)$ is the realization of τ with respect to G_y)

$$V[G_y] \models \tau \in A \Rightarrow V[G_y] \models \exists z (\Phi(x(y), z)).$$

Fix $z(y)$ such that $V[G_y] \models \Phi(x(y), z(y))$. By Lemma 2.9 (c), $z(y) \in M$.

$V[G_y]$ is a generic extension of V, using a set of conditions of cardinality $\leqq \mu < \kappa$. Hence by [13] T is still in $V[G_y]$, the tree associated with $\{x * z \mid \neg\Phi(x, z)\}$. Hence

$$V[G_y] \models T_{x(y)*z(y)} \text{ is well founded.}$$

Since $T \in M$ we conclude $M \models T_{x(y)*z(y)}$ is well founded. But $i(T) = T$ and in M, T is the tree associated with $\{x * z \mid \Phi(x, z)\}$,

$$M \models \exists z \Phi(x(y), z),$$

and we proved

$$M \models B \subseteq A.$$

□ Lemma 3.3

CONCLUSION OF THE PROOF OF THEOREM 3.2. In view of Lemma 3.3, M clearly satisfies

$$M \models \{x \mid \exists z \Phi(x, z)\} \text{ contains a perfect subset.}$$

But i is an elementary embedding of V into M which is the identity on all the parameters in Φ, hence

$$V \models A = \{x \mid \exists z \Phi(x, z)\} \text{ contains a perfect subset.}$$

□ Theorem 3.2.

4. Splitting ω_1

ω_1 is not measurable, hence whenever we are given an ω_1 complete ideal $\mathcal{I}$ on ω_1 we know that $\mathcal{I}$ is not a maximal ideal nor can it be extended to one. Hence ω_1 can be split into two disjoint sets in $\mathcal{P}(\mathcal{I})$. The problems dealt with in this section are motivated by an effort to determine the "simplest" such partition for a given ideal $\mathcal{I}$. We shall be mainly concerned with normal ideals on ω_1 (where an ideal on ω_1 is normal if it is normal as an ideal on $P_{\omega_1}(\omega_1)$ where we assume $P_{\omega_1}(\omega_1) - \omega_1 \in \mathcal{I}$). Note that for a precipitous ideal $\mathcal{I}$, $\mathcal{I}$ is normal iff for every generic filter in $\mathcal{P}(\mathcal{I})$, U, ω_1 is represented in $\mathrm{Ult}(V, \omega_1, U)$ by the identity function $I(\alpha) = \alpha$.

The minimal normal ideal is the ideal of nonstationary subset of ω_1, which is the dual to the filter generated by the closed unbounded subsets of ω_1. In order to describe the known results about partitioning ω_1 into two stationary sets we recall a few standard definitions. Let a be in ω^ω. We can consider a as coding a subset $\omega \times \omega$, hence as a relation on ω. a is called a code for the ordinal α if the relation coded by a is a well order relation of order type α.

DEFINITION 4.1. A subset of ω_1, A, is said to be Σ_n^1 (Π_n^1 respectively) if there exists a Σ_n^1 formula Φ (a Π_n^1 formula respectively) such that $\forall x$ (If x codes an ordinal α then $\Phi(x) \leftrightarrow \alpha \in A$).

Silver (unpublished) has shown that every Σ_1^1 subset of ω_1 (therefore every Π_1^1 subset of ω_1) either contains a closed unbounded set or its complement does. If for every real $x^{\#}$ exists (see Solovay [17]; this assumption follows from the existence of a measurable cardinal) then the same result holds for Σ_2^1 sets of countable ordinals. Harrington in [3] proved that, assuming the consistency of the existence of ineffable cardinals (which is consistencywise a much weaker assumption than $\forall x \exists x^{\#}$), it is consistent to assume that every Σ_n^1 subset of ω_1 either contains a closed unbounded subset or is disjoint from one.

If one wants to strengthen Harrington's result to get the consistency of the statement that every ordinal definable subset of ω_1 either contains or is disjoint from a closed unbounded subset, then one needs the consistency of a measurable cardinal. Conversely, from the consistency of a measurable cardinal one can get the consistency of "Every real ordinal definable subset of ω_1 either contains or is disjoint from a closed unbounded set" (see [7], section 22).

THEOREM 4.2. (M.C.) *Let $\mathcal{I}$ be a normal precipitous ideal on ω_1, then every Σ^1_3 set of countable ordinals either contains a closed unbounded subset or is in $\mathcal{I}$.*

(Note that by [6] if it is consistent to assume the existence of a normal precipitous ideal on ω_1, then it is consistent that the nonstationary ideal is precipitous.)

PROOF. First a lemma.

LEMMA 4.3. (M.C.) *Let $A \subseteq \omega_1$ be a Σ^1_3 set described by the Σ^1_3 formula Φ, i.e., $\alpha \in A$ iff $\forall x$ (if x codes α then $\Phi(x)$). Assume that $\mathcal{P}$ is a forcing notion of cardinality $< \kappa$ such that forcing with $\mathcal{P}$ introduces a real a coding ω_1 (in particular ω_1 becomes countable) such that $\Phi(a)$. Then A contains a closed unbounded set.*

PROOF. Let τ be a term in the forcing language for $\mathcal{P}$ such that it is forced that τ is a real coding ω_1 and $\Phi(\tau)$ holds. Let T be the tree associated with $\{x \mid \Phi(x)\}$. Since $|\mathcal{P}| < \kappa$, T is still the tree associated with $\{x \mid \Phi(x)\}$ in the generic extension $V[G]$. Hence it is forced that:

(I) τ is a real $\wedge$ T_τ is not well founded.

Since T_τ is not well founded we can find a term μ which is forced to be a branch in T_τ.

Consider the structure

$$\mathcal{L} = \langle R(\lambda), G, T, \omega_1, \mathcal{P}, \tau, \mu, \Vdash \rangle$$

for some λ which is large enough so that $T \in R(\lambda)$ and which is closed enough so that forcing of all the basic facts about τ and μ can be defined in it. (Among these basic we include statements like "τ codes ω_1", "μ is a branch in T_τ", etc.) By definiting Skolem functions for the structure $\mathcal{L}$ one can find an increasing sequence $\langle \mathcal{L}_\alpha \mid \alpha < \omega_1 \rangle$ of countable elementary substructures of $\mathcal{L}$ which is continuous, i.e., for α limit $\mathcal{L}_\alpha = \bigcup_{\beta<\alpha} \mathcal{L}_\beta$, and such that $\alpha \subseteq \mathcal{L}_\alpha$. Note that for all α, $\mathcal{L}_\alpha \cap \omega_1$ is a countable ordinal. Hence the set $B = \{\beta \mid \beta < \omega_1$, for some $\alpha < \omega_1$ $\beta = \mathcal{L}_\alpha \cap \omega_1\}$ is a closed unbounded subset of ω_1.

We claim that $B \subseteq A$. Let $\beta \in B$, hence $\beta = \mathscr{L}_\alpha \cap \omega_1$ for some α. Let M be the transitive isomorph of $\mathscr{L}_\alpha$; $h: \mathscr{L} \to M$ is the collapsing isomorphism. Clearly ω_1^M is β, $h(\mathscr{P})$ is a forcing notion in M, and $h(\tau)$, $h(\mu)$ are terms in the forcing language of $h(\mathscr{P})$ such that it is forced (over M) that $h(\tau)$ is a real, which codes $\omega_1^M = \beta$, and $h(\mu)$ is a branch in $h(T)_{h(\tau)}$. Now force over M with $h(\mathscr{P})$ to get a generic $h(\mathscr{P})$ filter G over M (note that M is countable). Therefore we assume $G \in V$. The realization of $h(\tau)$ with respect to G gives a real x which clearly codes β, and the realization of $h(\mu)$ is a branch in $h(T)_x$, f.

It is now obvious that the set $\{h^{-1}(t) \mid t \in f\}$ is a branch in T_x. (Note that h^{-1} is the identity on elements of $\mathrm{Seq}(\omega)$.) Hence, we get

$$T_x \text{ has an infinite branch.}$$

Hence we get that $\Phi(x)$ holds, but by definition of A we get $\beta \in A$ since x codes β. Thus we have proved $B \subseteq A$ and A contains a closed unbounded set.

□ Lemma 4.3

Conclusion of the Proof of Theorem 4.2. Given a Σ_3^1 set of countable ordinals described as the sets of ordinals whose codes satisfy the Σ_3^1 formula Φ, assume A is not in $\mathscr{I}$. We shall produce a forcing notion which will satisfy all the assumptions of Lemma 4.3, from which we shall be able to conclude that A contains a closed unbounded subset of ω_1.

The forcing notion which satisfies the assumption of Lemma 4.3 is simply $\mathscr{P} = \mathscr{P}(\mathscr{I} \restriction A)$ where $\mathscr{I} \restriction A$ is the ideal defined by $B \in \mathscr{I} \restriction A$ iff $B \cap A \in \mathscr{I}$. Forcing with $\mathscr{P}$ is simply forcing with $\mathscr{P}(\mathscr{I})$ except that we start by putting A in the generic filter. Hence a generic filter in $\mathscr{P}$ is a generic filter in $\mathscr{P}(\mathscr{I})$. Denote such a filter by U. Let I be the identity function on ω_1, i.e., $I(\alpha) = \alpha$ for all $\alpha < \omega_1$. As usual let M be the transitive isomorph of $\mathrm{Ult}(V, \omega_1, U)$ and $i: V \to M$ the natural embedding. By normality $[I]_U = \omega_1$.

Since $A \in U$, $\{\alpha \mid I(\alpha) \in A\} \in U$, hence by Łos theorem $\omega_1 \in i(A)$. Note that ω_1 is countable in M. Pick any code for ω_1 in M and call this code x. By definition of $i(A)$ in M

$$M \models \Phi(x). \tag{II}$$

(i is the identity on parameters of Φ.)

Let T be the tree associated with $\{x \mid \Phi(x)\}$. Since $|\mathscr{P}| < \kappa$, T is still the tree associated with this set in $V[U]$ and by Lemma 1.12 also in M. Hence we get from (II) that

$$M \models T_x \text{ is not well founded.}$$

Therefore

$$V[U] = T_x \text{ is not well founded,}$$

which implies

$$V[U] \models \Phi(x).$$

Hence we get a forcing extension which introduces a code for ω_1 which satisfies Φ. By Lemma 4.3 A contains a closed unbounded set. □ Theorem 4.2

The result for Σ_4^1 set of ordinals (which is the next theorem) is weaker. To state it we have to introduce the following notation.

Let $\mathcal{I}$ be a normal ideal on $P_{\omega_1}(\lambda)$ for some λ, then this ideal naturally induces a normal ideal on ω_1 by the definition (for $A \subseteq \omega_1$)

$$A \in \mathcal{I}_{\omega_1} \quad \text{iff} \quad \{P \mid P \in P_{\omega_1}(\lambda), P \cap \omega_1 \in A\} \in \mathcal{I}.$$

It is easy to check that $\mathcal{I}_{\omega_1}$ is a normal ideal on ω_1. Note that for the usual constructions of precipitous ideals, if $\mathcal{I}$ is precipitous then $\mathcal{I}_{\omega_1}$ is precipitous but this does not have to hold in general.

THEOREM 4.4. (M.C.) *Let $\mathcal{I}$ be a normal precipitous ideal on ω_1 and let $\mathcal{J}$ be a normal precipitous ideal on $P_{\omega_1}(2^{2^{\omega_1}})$, then every Σ_4^1 subset of ω_1 is either in $\mathcal{I}$ or its complement in $\mathcal{J}_{\omega_1}$.*

(*Note*: By the remark preceding the theorem we get for the usual $\mathcal{J}$'s that the set is in $\mathcal{J}_{\omega_1}$ or its complement is, since $\mathcal{J}_{\omega_1}$ is precipitous.)

PROOF. Let $A \subseteq \omega_1$ be a Σ_4^1 set such that it is the set of all ordinals having codes x satisfying $\exists z \Phi(x, z)$ where Φ is Π_3^1. Let T be the tree associated with $\{x * z \mid \neg\Phi(x, z)\}$. Assume that neither A is in $\mathcal{I}$ not $\omega_1 - A$ in $\mathcal{I}_{\omega_1}$. The fact that $\omega_1 - A$ is not in $\mathcal{I}_{\omega_1}$, implies that the set

$$\text{(III)} \qquad A^* = \{P \mid P \cap \omega_1 \in \omega_1 - A, P \in P_{\omega_1}(2^{2^{\omega_1}})\} \notin \mathcal{I}.$$

We can find a generic filter in $\mathcal{P}(\mathcal{I})$, U, such that $A^* \in U$. As usual M is the transitive isomorph of $\mathrm{Ult}(V, P_\omega(2^{2^{\omega_1}}), U)$, i the embedding of V into M. By Lemma 2.6 the function f defined by $f(P) =$ The order type of $P \cap \omega_1$ satisfies $[f]_U = \omega_1$.

Since by normality of $\mathcal{J}$ we get that

$$\{P \mid P \cap \omega_1 \notin \omega_1, P \in P_{\omega_1}(2^{2^{\omega_1}})\} \in \mathcal{J},$$

hence since $\mathcal{U} \subseteq \mathcal{P}(\mathcal{J})$ we know $f(P) = P \cap \omega_1$ is a set in U. But Łos theorem, (III), and the fact that $A \in U$ imply that

$$M \models \omega_1 \in i(\omega_1 - A) = i(\omega_1) - i(A).$$

Hence by definition of A applied to $i(A)$ in M

(IV) $$M \models \forall x \text{ (If } x \text{ is a code for } \omega_1 \text{ then } \neg \exists z \Phi(x, z)).$$

We are going to get a contradiction by producing in M a code for ω_1, x, for which $\exists z \Phi(x, z)$ holds in M.

Remember that we assume $A \not\in \mathcal{I}$. Now use Lemma 2.9 to get in M a $\mathcal{P}(\mathcal{I})$ generic filter over V, G such that $A \in G$. ($\mathcal{I}$ is ideal on ω_1, hence the cardinality of the family of dense subsets of $\mathcal{P}(\mathcal{I})$ is at most $2^{2^{\omega_1}}$.) Note also that every real of $V[G]$ is in M. Let N be the isomorph of $\mathrm{Ult}(V, \omega_1, G)$ and j the elementary embedding of V into N.

By normality of $\mathcal{I}$, ω_1 is represented in $\mathrm{Ult}(V, \omega_1, G)$ by the identity function ($I(\alpha) = \alpha$). Therefore since $\{\alpha \mid I(\alpha) \in A\} \in U$ we get $\omega_1 \in i(A)$. By definition of A we get

(V) $$N \models \exists x (x \text{ codes } \omega_1 \text{ and } \exists z \Phi(x, z)).$$

Fix such $x \in N$ and $z \in N$ satisfying $\Phi(x, z)$. Since $N \subseteq V[G]$ and since every real in $V[G]$ is in M, we get $x, z \in M$.

By Lemma 1.2, $j(T) = T$, and T is in N the tree associated with the Σ^1_3 set $B = \{x * z \mid \neg\ \Phi(x, z)\}$. By (V)

$$N \models T_{x*z} \text{ is well founded.}$$

Similarly by Lemma 1.4, $i(T) = T$, and T is in M the tree associated with B. Hence $T \in M$ and since $x, z \in M$

$$M \models T_{x*z} \text{ is well founded,}$$

$$M \models \Phi(x, z) \Rightarrow M \models \exists z \Phi(x, z).$$

Therefore we have found in M a code for ω_1 (x still codes ω_1 in M) which satisfies the Σ^1_4 formula $\exists z \Phi(x, z)$. We have a contradiction to (IV).

□ Theorem 4.4

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